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Monte Carlo Simulation for Reliability and Availability Analysis

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Some Examples:

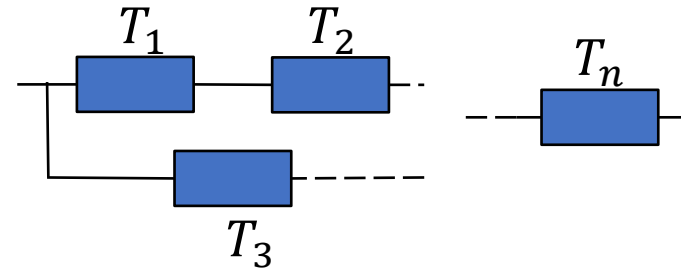
1) Unreliability @mission time t_M :

$$F_T(t_{miss}) = P\{T \leq t_{miss}\} = P\{q(T_1, \dots, T_n) \leq t_{miss}\} = \\ = P\{q(T_1, \dots, T_n) \leq t_{miss}\} = \int_{(t_1, \dots, t_n): q(t_1, \dots, t_n) \leq t_{miss}} f_{T_1, \dots, T_n}(t_1, \dots, t_n) dt_1 \dots dt_n$$

- T_i = Failure time of component i
- $T = q(T_1, \dots, T_n)$ = System failure time

Examples for q :

- Series system $\rightarrow q(T_1, \dots, T_n) = \min(T_1, \dots, T_n)$
- Parallel system $\rightarrow q(T_1, \dots, T_n) = \max(T_1, \dots, T_n)$



2) $MTTF = \int_0^{+\infty} t f_T(t) dt$

MC Estimation of Definite Integrals

$$G = \int_a^b h(x) dx = \int_a^b g(x) f_X(x) dx$$



- X is a random variable with pdf $f_X(x)$:
 - $g(x)$ is a random variable
- $$\begin{cases} f_X(x) \geq 0 \\ \int_a^b f_X(x) dx = 1 \end{cases}$$



$$E[g(x)] = \int_a^b g(x) f_X(x) dx = G$$

Problem → Estimate $E[g(x)]$

Solution → Dart Game

- 1) for $i = 1, 2, \dots, N$
 - Sample X_i from $f_X(x)$
 - Compute $g(X_i)$End

- the probability that a shot X_i hits $[x, x + dx]$ is $f(x)dx$
- the award is $g(X_i)$

2) Compute

$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \bar{g}$$

Random variable!

$\swarrow \searrow$
 $Var[G_N] \quad E[G_N]$



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Random variable!

\swarrow \searrow
 $\text{Var}[G_N]$ $E[G_N]$

Is G_N a good estimator of $E[g(x)]$?

Characteristics of a good estimator:

- Unbiased $\rightarrow E[G_N] = G$
- Consistent $\rightarrow \lim_{N \rightarrow +\infty} E[(G_N - G)^2] = 0$

MC Evaluation of Definite Integrals (1D): Why G_N is a good estimator of G ?

$$G_N = \frac{1}{N} \sum_{i=1}^N g(x_i)$$



G_N is a random variable with:

$$E[G_N] = E\left[\frac{1}{N} \sum_{i=1}^N g(X_i)\right] = \frac{1}{N} \sum_{i=1}^N E[g(X_i)] = \frac{1}{N} \sum_{i=1}^N E[g(x)] = G$$

$$\text{Var}[G_N] = \text{Var}\left[\frac{1}{N} \sum_{i=1}^N g(X_i)\right] = \frac{1}{N^2} \sum_{i=1}^N \text{Var}[g(X_i)] = \frac{1}{N} \text{Var}[g(x)]$$



G_N is an unbiased estimator of G :

$$E[G_N] = G$$

G_N is a consistent estimator of G :

$$\lim_{N \rightarrow \infty} \text{Var}[G_N] = 0$$

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi} = 0.6366$$

How can we write the integral for
MC estimation?

$$f(x) = ?$$

$$g(x) = ?$$

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi} = 0.6366$$

By setting:

$$f(x) = 1 \quad \text{for } x \in [0,1]$$

$$g(x) = \cos\left(\frac{\pi}{2}x\right)$$

We perform $N=10^4$ trials:

$$\begin{array}{l} X_i \rightarrow U[0,1) \\ g(x_i) = \cos\left(\frac{\pi}{2}X_i\right) \end{array} \quad \Rightarrow \quad G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$

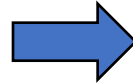
MC Evaluation of Definite Integrals (1D)

Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$$

We perform $N=10^4$ trials:

$$\begin{aligned} x_i &\rightarrow U[0,1) \\ g(x_i) &= \cos\left(\frac{\pi}{2}X_i\right) \\ N &= 10^4 \end{aligned}$$



$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$



$$\text{Var}[G_N] = \frac{1}{N} \text{Var}[g(x)] = \frac{1}{N} \left(E[g^2(x)] - (E[g(x)])^2 \right) = \frac{1}{N} \left(E[g^2(x)] - G^2 \right)$$

Unknown in a practical case!

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$$

We perform $N=10^4$ trials:

$$x_i \rightarrow U[0,1]$$
$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$
$$N = 10^4$$

→

$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$



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$$G = E[g(x)] = \frac{2}{\pi}$$

$$E[g^2(x)] = \int_0^1 \cos^2\left(\frac{\pi}{2}x\right) dx = \frac{1}{2}$$

Unknown in a practical case!

$$\text{Var}[G_N] = \frac{1}{10^4} \left(\frac{1}{2} - \left(\frac{2}{\pi}\right)^2 \right) = 9.47 \cdot 10^{-6}$$

MC Evaluation of Definite Integrals (1D) Example

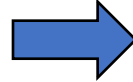
$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$$

We perform $N=10^4$ trials:

$$x_i \rightarrow U[0,1]$$

$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$

$$N = 10^4$$



$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0.6342$$



$$\text{Var}[G_N] = \frac{1}{N} \text{Var}[g(x)] = \frac{1}{N} \left(E[g^2(x)] - (E[g(x)])^2 \right) = \frac{1}{N} \left(E[g^2(x)] - G^2 \right)$$



$$G \approx G_N = 0.6342$$

$$E[g^2(x)] \approx \frac{1}{N} \sum_{i=1}^N g^2(X_i)$$

They can be computed during the MC simulation!



$$\text{Var}[G_N] \approx \frac{1}{N} \left(\overline{g^2} - G_N^2 \right) = 9.6 \cdot 10^{-6}$$

Estimated Variance!



$$G = 0.6342 \pm \sqrt{9.6 \cdot 10^{-6}} = 0.6342 \pm 0.0031$$

True value is 0.6366

CENTRAL LIMIT THEOREM:

For large N , the distribution of G_N converges to a Gaussian distribution with mean G and variance $\sigma_{G_N}^2 = \frac{1}{N} \text{Var}[g(x)]$:

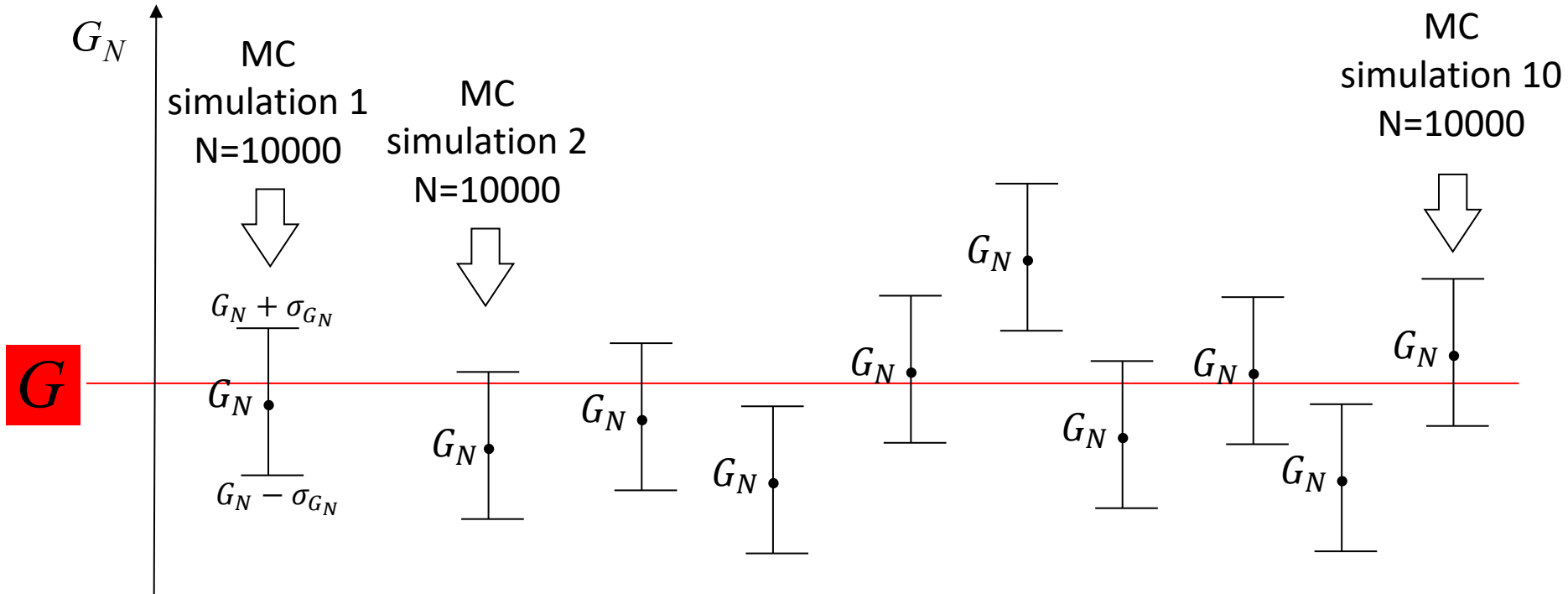
$$N \rightarrow +\infty \Rightarrow G_N \sim N(G, \sigma_{G_N}^2)$$

KNOWING THAT, FOR A GENERIC GAUSSIAN DISTRIBUTION $N(\mu, \sigma^2)$:

$$P(x \in [\mu - k \cdot \sigma, \mu + k \cdot \sigma]) = \int_{\mu - k \cdot \sigma}^{\mu + k \cdot \sigma} N(x; \mu, \sigma^2) dx \approx \begin{cases} 0.683 & (\text{for } k = 1) \\ 0.954 & (\text{for } k = 2) \\ 0.997 & (\text{for } k = 3) \end{cases}$$

IT FOLLOWS THAT:

$$P(G_N \in [G - k \cdot \sigma_{G_N}, G + k \cdot \sigma_{G_N}]) \approx \begin{cases} 0.683 & (\text{for } k = 1) \\ 0.954 & (\text{for } k = 2) \\ 0.997 & (\text{for } k = 3) \end{cases}$$



unknown

$$P(G_N \in [G - \sigma_{G_N}; G + \sigma_{G_N}]) \approx 0.68$$

$$P(G \in [G_N - \sigma_{G_N}; G_N + \sigma_{G_N}]) \approx 0.68$$

- Why Monte Carlo instead of deterministic numerical integration?

Because the latter suffers from two major issues when dealing with highly multidimensional problems:

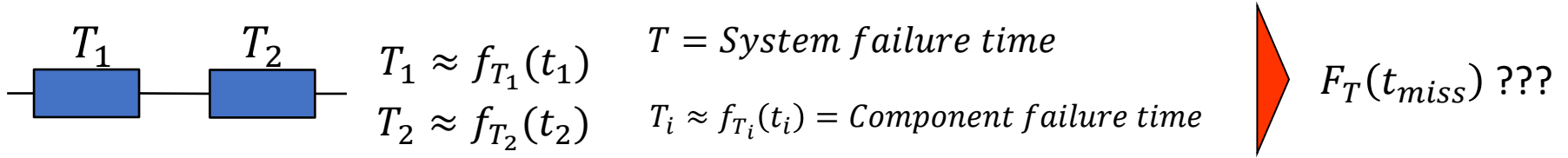
1. The number of function evaluations (grid) increases combinatorially with the number of dimensions
2. The boundaries of the multidimensional integration domain D become intractable

- The estimate G_N becomes more precise (less uncertain) as the estimator variance $Var[G_N]$ decreases!
- How can we achieve lower $Var[G_N] = \frac{1}{N} Var[g(x)]$?
 1. Increasing the number N of MC trials \Rightarrow “brute force”
 2. Decreasing $Var[g(x)] \Rightarrow$ variance reduction techniques

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx \equiv \int_D g_1(x) f_1(x) dx$$

Forced (biased) MC simulation

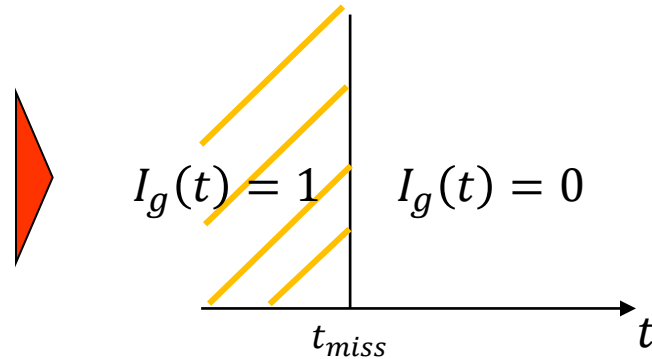
Some Examples of MC Estimation of RAM quantities of Interest



$$F_T(t_{miss}) = P\{T \leq t_{miss}\} = \int_0^{t_{miss}} f_T(t) dt = \int_0^{+\infty} I_g(t) f_T(t) dt$$

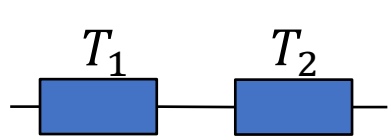
with:

$$I_g(t) = \begin{cases} 1 & \text{if } t \leq t_{miss} \\ 0 & \text{otherwise} \end{cases}$$



$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

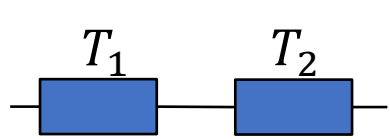
$$G = F_T(t_{miss}) \quad g(x) = I_g(t) \quad f(x) = f_T(t)$$



$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$
$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

$$t_{miss} = 8760 \text{ h}$$
$$\lambda_1 = 2 \cdot 10^{-4} \text{ h}^{-1}$$
$$\lambda_2 = 5 \cdot 10^{-3} \text{ h}^{-1}$$

OBJECTIVE:
MC Estimate System Unreliability
at the Mission Time



$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$

$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

$$t_{miss} = 8760 \text{ h}$$

$$\lambda_1 = 2 \cdot 10^{-4} \text{ h}^{-1}$$

$$\lambda_2 = 5 \cdot 10^{-3} \text{ h}^{-1}$$

OBJECTIVE:

MC Estimate System Unreliability
at the Mission Time

$$N = 10000 \rightarrow \lambda_{sys} = \lambda_1 + \lambda_2 \rightarrow \text{for } i = 1, \dots, N \rightarrow t_i = -\frac{1}{\lambda_{sys}} \ln(1 - r_i), r_i \rightarrow U[0,1)$$

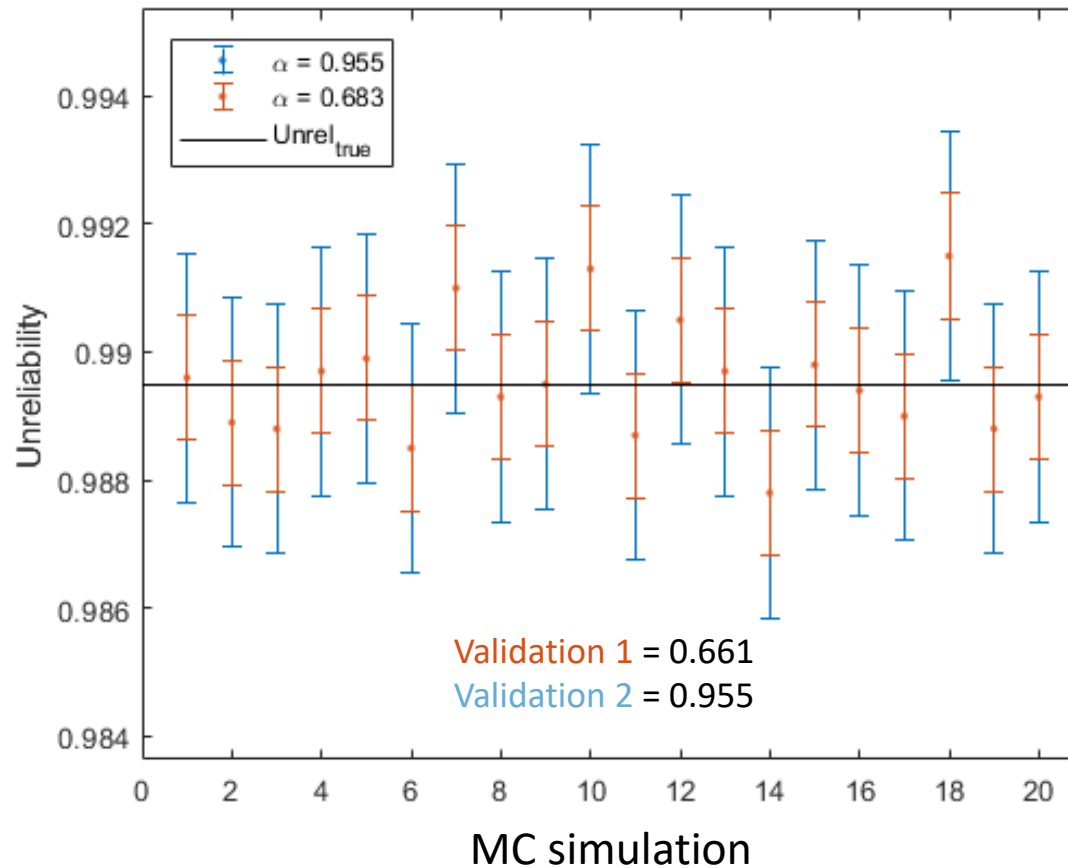
$$F_N(t_{miss}) = \frac{1}{N} \sum_{i=1}^N I_g(t_i) = 0,9891 \text{ where } I_g(t_i) = \begin{cases} 1 & \text{if } t_i < t_{miss} \\ 0 & \text{otherwise} \end{cases}$$

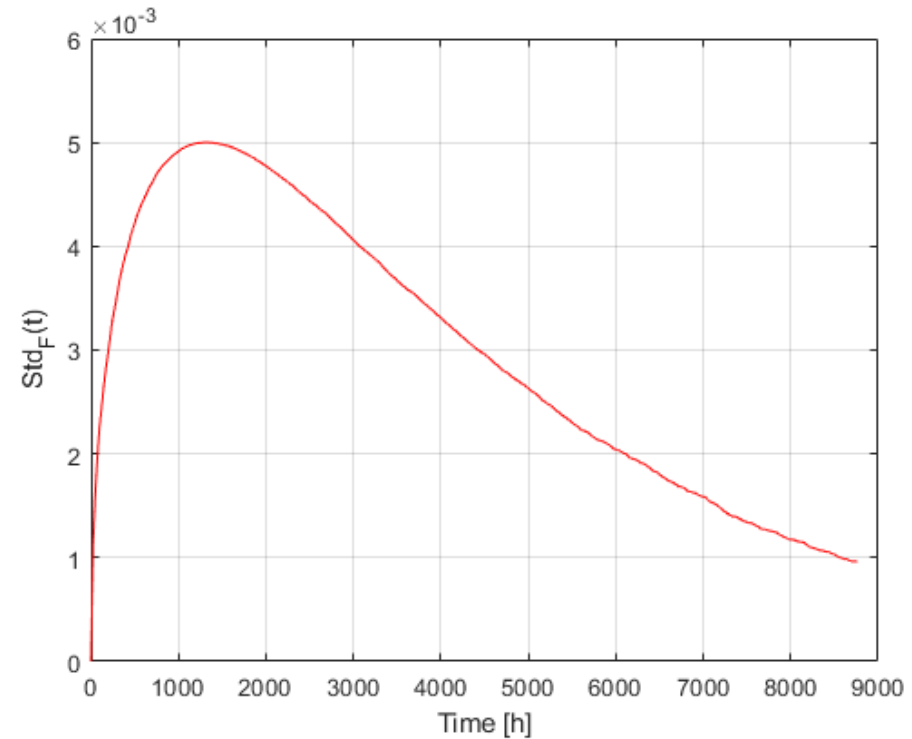
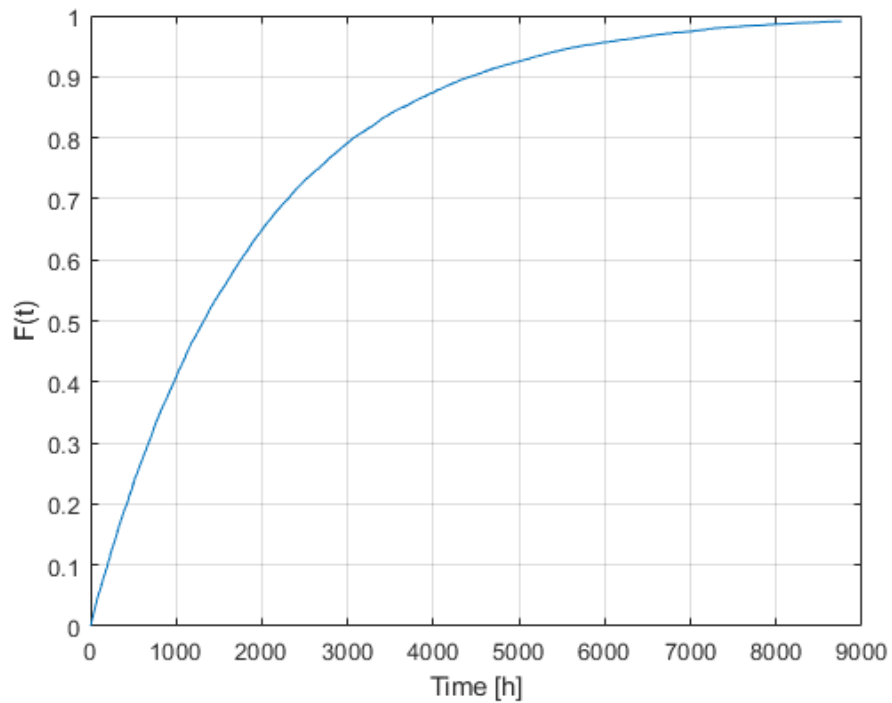
$$Var[F_N(t_{miss})] \approx \frac{1}{N} \left(\left(\frac{1}{N} \sum_{i=1}^N (I_g(t_i))^2 \right) - F_N^2(t_{miss}) \right) \approx \frac{1}{N} (F_N(t_{miss}) - F_N^2(t_{miss})) = 1,08 \cdot 10^{-6}$$

$$\text{MC ESTIMATION OF SYSTEM UNRELIABILITY} = F_N(t_{miss}) \pm \sqrt{Var[F_N(t_{miss})]} = 0,9891 \pm 1,0 \cdot 10^{-3}$$

$$\text{TRUE VALUE OF SYSTEM UNRELIABILITY} = 1 - e^{-(\lambda_1 + \lambda_2)t_{miss}} = 0,9895$$

Repeating the system unreliability estimation 1000 times ...





$$MTTF = \int_0^{+\infty} t f_T(t) dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

$$G = MTTF \quad g(x) = t \quad f(x) = f_T(t)$$

Exponential failure time T



$$f_T(t) = \lambda e^{-\lambda t}$$

$$\lambda = 0,2 \text{ h}^{-1}$$

OBJECTIVE:
MC Estimate System MTTF

$$MTTF = \int_0^{+\infty} t f_T(t) dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

$$G = MTTF \quad g(x) = t \quad f(x) = f_T(t)$$

Exponential failure time T



$$f_T(t) = \lambda e^{-\lambda t}$$

$$\lambda = 0,2 \text{ h}^{-1}$$

OBJECTIVE:
MC Estimate System MTTF

Considering $N = 10000$ trials:

$$MTTF_N = \frac{1}{N} \sum_{i=1}^N T_i = 4,98 \text{ h}$$

$$Var[MTTF_N] \approx \frac{1}{N} \left(\left(\frac{1}{N} \sum_{i=1}^N T_i^2 \right) - MTTF_N^2 \right) = 0,0024 \text{ h}$$

$$\text{MC ESTIMATION OF SYSTEM } MTTF = MTTF_N \pm \sqrt{Var[MTTF_N]} = 4,98 \pm 0,049$$

$$\text{TRUE VALUE OF SYSTEM } MTTF = \frac{1}{\lambda} = 5 \text{ h}$$

Repeating the system MTTF estimation 1000 times ...

